a **vector space** is a set whose elements are *vectors*, can be added together and multiplied by a scalar. a **vector space** is also called a **linear space.**

It is important to realize that a vector space consists of four entities:

1. A set V of vectors.
2. A set of scalars. In this class, it will always be the set of real numbers R. (Later on, this could be the set of complex numbers C.)
3. A vector addition denoted by +.
4. A scalar multiplication

The set of all d~x defines the tangent space at x. By assigning a tangent vector to every spacetime point, we can recover the usual concept of a vector field. However, without additional preparation (tangent space shit) one cannot compare vectors at different spacetime points because they lie in different tangent spaces.

set of all vectors is vector space?

A one-form is defined as a linear scalar function of a vector. Takes vector and gives number



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There are 2 vector spaces

1. Dual vector space === The vector space of one-forms is called the cotangent vector space (set of all one-form functions?)
2. The linear space of vectors (tangent space).

Set of all one-forms is vector space

The set of all one-forms is a complementary vector spaces to the linear vector space of vectors.

The scalar product notation with subscripts makes this clearer 

the fundamental object in quantum mechanics is the state vector, represented by a ket |ψi

in linear vector space (Hilbert space). (set of vectors)

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| Notes: |
| Graphical user interface, text, application, email  Description automatically generated  The dot product of 2 function is:  enter image description here |

In quantum mechanics:

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in quantum mechanics is the state vector, is a ket (vector 1)

in a linear vector space (Hilbert space) (set of all vectors?)

A distinct Hilbert space (set of all vectors?) is given by the set of bra vectors

Bra vectors and ket vectors are linear scalar functions of each other.

The scalar product <φ|ψ> maps a bra vector and a ket vector to a scalar called a probability amplitude.

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| A tensor of rank (m, n),  also called a (m, n) tensor  is a scalar function of  m one-forms and n vectors |

It follows at once that scalars are tensors of rank (0, 0), vectors are tensors of rank (1, 0) and one-forms are tensors of rank (0, 1).  
The scalar product is a tensor of rank (1, 1), which we will denote I and call the identity tensor:

the tensor (or outer) product, combines two tensors of ranks (m1, n1) and (m2, n2) to form a tensor of rank (m1 + m2, n1 + n2) by simply combining the argument lists of the two tensors thereby expanding the dimensionality of the tensor space.



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a linear combination (addition) of tensors of rank (m, n) is also a tensor of rank (m, n),

Tensor is non-commutative: A ⊗B not equal B⊗A

There are three ways to change the rank of a tensor: tensor (outer) product, contraction, and gradients

the product of two vectors A and B gives a tensor of 2 one-forms and no vectors

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the outer product of two coordinate vectors is a matrix. If the two vectors have dimensions n and m, then their outer product is an n × m matrix. More generally, given two tensors, their outer product is a tensor

Metric tensor: Is it possible to obtain a scalar from two vectors or two one-forms?

1. The scalar product A picture containing text, watch, clock, gauge

   Description automatically generated requires a vector and a one-form.

Is it possible to obtain a scalar from two vectors or two one-forms? YES

1. Any tensor of rank (0, 2) will give a scalar from two vectors e.g.: g
2. Any tensor of rank (2, 0) combines two one-forms to give a scalar e.g.: g-1

However, there is a special (0, 2) tensor field g X called the metric tensor

and a related (2, 0) tensor field g −1 X called the inverse metric tensor

The metric tensor g is a scalar function of two vectors. It gives the dot product of vectors V and W

Similarly, g-1 returns a scalar from two one-forms P˜ and Q˜, also called the dot product:

But we reserve the dot product notation for the metric and inverse metric tensors just as we reserve the angle brackets scalar product notation for the identity tensor



Question: we know that any tensor of the rank (0,2) takes 2 vectors and give scalar, but the metric tensor is only the (“” function””) of that rank that give the dot product???

To simplify calculations, it is helpful to introduce a set of linearly independent basis vector and one-form fields spanning our vector and dual (set of all one-forms?) vector spaces.

(Set of all vectors?) (And set of all one-forms?)

Note: Vector Span

In general, if you're thinking about a vector on its own, think of it as an arrow,

if you're dealing with a collection of vectors, it's convenient to think of them all as points.

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| the span of most pairs of vectors ends up being the entire infinite sheet of two-dimensional space, but if they line up, their span is just a line. |

The idea of span gets a lot more interesting if we start thinking about vectors in three-dimensional space. For example, if you take two vectors, in 3-D space, that are not pointing in the same direction, what does it mean to take their span?

Well, their span is the collection of all possible linear combinations of those two vectors, meaning all possible vectors you get by scaling each of the two of them in some way, and then adding them together. You can kind of imagine turning two different knobs to

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| change the two scalars defining the linear combination, adding the That tip will trace out some kind of flat sheet, cutting through the origin of three-dimensional space. This flat sheet is the span of the two vectors, |

So, what happens if we add a third vector and consider the span of all three of those guys? linear combination of three vectors is defined pretty much the same way as it is for two, you'll choose three different scalars, scale each of those vectors, and then add them altogether. And again, the span of these vectors is the set of all possible linear combinations. Two different things could happen here:

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| If your third vector happens to be sitting on the span of the first two  then the span doesn't change; you're sort of trapped on that same flat sheet. |

In other words, adding a scaled version of that third vector to the linear combination

doesn't really give you access to any new vectors.

But if you just randomly choose a third vector, it's almost certainly not sitting on the span of those first two.

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| Then, since the third vector is pointing in a separate direction, it unlocks access to every possible three-dimensional vector. One way I like to think about this is that as you scale that new third vector, vector moves around that span sheet of the first two, sweeping it through all of space. |

In the same way, practical calculations in quantum mechanics often start by expanding the ket vector in a set of basis kets, e.g., energy eigenstates.

the dimensionality of spacetime (four) equals the number of linearly independent basis vectors and one-forms

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| there are four functions of that for each point as well as the 4d vector  each function takes vector and give a scaler |

We have vector field (set of all combination of vectors) for each point in space, but we can’t compare the vectors of different points because they lie in different tangent space

The vector field ==== 

where µ labels the basis vector (e.g., µ = 0, 1, 2, 3) and x labels the spacetime point.

Any four linearly independent basis vectors at each spacetime point will work.



Given a basis, we may expand any  as a linear combination of basis vectors:



For now, we consider only tangent vectors at x. To emphasize the status of a tangent vector, we will occasionally use a subscript notation: .

By assigning a tangent vector to every spacetime point x, we can recover the usual concept of a vector field set of vectors

The span of a set of vectors is the set of all linear combinations of these vectors.

However, without additional preparation one cannot compare vectors at different spacetime points because they lie in different tangent spaces.

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The dual basis is an example of one-form basis, but it is a special one

Any four linearly independent functions are a one-form basis and there are four functions of that for each point

Dual vector space === The vector space of one-forms is called the cotangent vector space (set of all one-form functions?)

* g is a (0, 2) tensor (function takes 2 vectors and give a scaler)
* g −1 is (2, 0) tensor (function takes 2 one-forms and give a scaler)

that tensor is called a metric and its special because:



that allows us to convert vectors to one-forms because there are several equivalent ways to obtain scalars from vectors V~ and W~ and their associated one-forms V→ and W→

Although:



We have 



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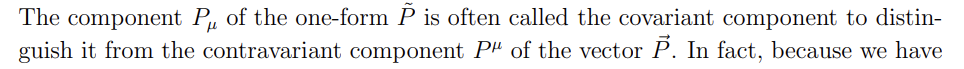


Equation (13) is a system of four linear equations ˜e µ = ˜e1 ˜e2 ˜e3 ˜e4 at each spacetime point

and it has a unique solution. (The reader may show that any nontrivial transformation of the dual basis one-forms will violate eq. 13.) Now we may expand any one-form field P˜X in the basis of one-forms:

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to get the components of vectors or the components of one-forms, using the fact that vectors are scalar functions of one-forms and vice versa. 🡺 evaluates the vector using the appropriate basis one-form:

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Evaluate the vector A using the appropriate function eµ where e.g., µ = 0, 1, 2, 3)



A of e- µ is scaler because e- µ is a scaler function by the definition of one-forms

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This are the scaler values of the three special cases tensor (identity, metric, inverse metric)

The identity equation is based on

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Basis vectors and one-forms’’ µ’’ allow us to represent any tensor equations

using components. µ for example, the dot product between two vectors or two one-forms and the scalar product between a one-form and a vector may be written using components as



The scalar product A picture containing text, watch, clock, gauge

Description automatically generated requires a vector and a one-form.

Is it possible to obtain a scalar from two vectors or two one-forms? YES

Any tensor of rank (0, 2) will give a scalar from two vectors e.g.: g

Any tensor of rank (2, 0) combines two one-forms to give a scalar e.g.: g-1

A→ = Aµ e→µ

AGAIN:



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 the rank (1,1) takes a Tilda and an arrow and gives scaler

tensor product of two vectors A and B gives a rank (2, 0) tensor

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The tensor outer product of two vectors is a function (tensor) that takes two function (one-forms) and gives a scaler

 of the rank (2,1)



g is metric and takes two victors and gives scaler (the dot product) at mu and nu

g-1 takes two functions and gives the dot product of them at mu an nu



So 

g is tensor of rank (0,2)

g-mu-nu is the dot product of vectors: e-mu and e-nu (the metric tensor that gives that product and its rank is 0,2)

the outer product of e-Tilda-mu and e-Tilda-nu is tensor of rank (0,2)

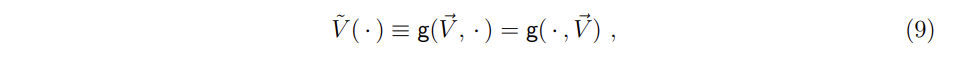
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 is the dot product of them so  is the dot product of  and the basis vector  while  so  = 

 is a function that needs a vector to give the dot product of it with  while  is like it but it has found the vector  (basis vector) which it will give its dot product with 

Rank-2 Tensors (such as the Einstein tensor) are represented as matrices, two dimensional grids of numbers. The simplest metric (the Euclidean metric in xyz xyz coordinates) simply looks like: Icon

Description automatically generated it’s a metric tensor gives the scaler dot product of the vectors that are its arguments

contraction of a tensor.

the (1, 3) tensor R, may be contracted on its second vector argument to give a (0, 2)

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The left side is the contracted (shorter arguments) (0 Tilda ,2 arrows)

Right side is (1,3) The arguments are replaced by basis vectors  and  and summation from 0 to 3 (4 dimensions)

Ricci tensor (0,2) is contracted defined by this equation form the Riemann curvature tensor of rank (1,3)

3.4 Change of basis ==== transformation, may be inverted: any four-by-four matrix may be used for Λ ν µ′. ... scalar products of the old and new basis vectorsWord

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That is, any nonsingular four-by-four matrix may be used for Λ ν µ′. The transformation is not restricted to being a Lorentz transformation; the local reference frames defined by the bases are not required to be inertial (or, in the presence of gravity, freely falling). Because the transformation matrix is assumed to be nonsingular, the transformation may be inverted:

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This makes it clear that A arrow is a mathematical entity (vector) that is invariant to the coordinates (basis) (same for nu and mu-prime) and to get mu-prime transform the nu by the change of basis transformation matrix

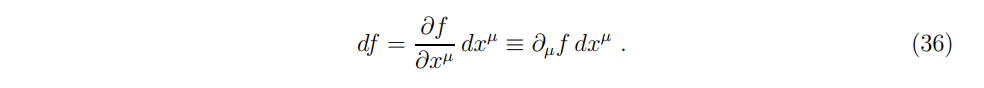
The two matrices for tensor transformation: L, L-1

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we introduce one more idea: a coordinate system. A coordinate system is simply a set of four differentiable scalar fields x µ X (not one vector field — note that µ labels the coordinates and not vector components)

Consider any scalar field fX. Treating f as a function of the coordinates, the difference in f between two infinitesimally close points is



Equation (36) may be taken as the definition of the components of the gradient… Gradient of function that takes several scalers input point p is and gives another function that takes vectors that is at the point p are the partial derivatives off at p

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Gradient of function that takes several scalers input point p is and gives another function that takes vectors that is at the point p are the partial derivatives of f at p

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* Note: The gradient (nebula notation) of a function that takes 2 input and gives output is another function that takes 2 input and gives out 2 outputs, so the gradient is another operator that takes a function and gives a function, but the new vector of functions has higher number of outputs as every output (dimension of this vector) is the partial derivative of the original function with respect to one input.
* Note: Directional derivative of a function = dot product of a vector by its gradient vector.

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rewrite equation (36) in the coordinate-free manner A picture containing shape

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df is sclaer gradient is one-form and dx is a vector

equation 36 inclues the coordinate mu where the one-form component of the gradient is the partial drivative of f with respect to mu where the vector is dx at mu

But what is a tensor though? Lol.

a vector has contra-variant components and co-variant components

the contravairant component is the one we usually discribe the vector with, they are the scaler that we multiply the basis vectors with if the basis vector changes the scalers will also change hence “contra” we use superscripts V1 V2 V3 where 1,2,3 are the index values if the basis vector (the coordinate system) was transformed by L the magnitude of the component will be transformed by L-1  the effects of L and L-1 will cancel each other to give us the same vector or tensor

Note #1: Transformations or matrixes that preserve the dot product, preserve the angels in a manner that no stretching or squishing is taking place. That why the projection part of the dot product calculation holds and the parallel lines still parallel those has special name orthonormal and they are just rotational transformation changing positions without stretching. Rotational matrixes

Note #2: To switch to another coordinate system, we just use a matrix (A)and to switch back to the original coordinate system we use the inverse of this matrix(A -1 ). So, if we have a matrix describe certain transformation in certain coordinate system(M). And we need to know what is the matrix that do the same transformation in the other coordinate system we apply the expression (A -1 ) (M) (A).

if we discribe the vector as the dot product with the basis vector, when the basis vector changes the dot product will change accordingly hense “co-variant” we use subsript V1 V2 V3

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if we have 2 vectors V and P each are 3d, when “” multiplied”’ we get 3 by 3 matrix, since the index values are superscripts then these are contra-variant

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The last one is a 3D tensor of rank 3

Note: For some transformations (matrixes) there is some vector that maintains the same span before and after the transformation (they might be stretched or squished but not rotated) so that they will land on the same span. And any vector on this span (in case of 2-d matrix this span is just an infinite line) will also still remain on the span line. Those vectors are named eigenvectors and the factor my which they are scaled is called the eigenvalue. Consider a rotational matrix in 3d space, where do you think its eigenvectors are? They are the axis of rotation. And what is the eigenvalue for those vectors? It is 1 because rotation don’t stretch anything it just rotate the space so that only the axis of rotation maintains its span and its scaler is always 1. One the other hand, 2d oration has no eigenvalue (everything is just rotating no vector will maintain its span). In general case, this is donated by the expression: AV = λV where the A is the matrix and lambda is the scaler (eigenvalue) and V is the eigen vector

To solve for eigenvalue and eigenvector is pretty strait forward. lambda is just multiplied with the identity matrix (whose diagonal values are ones and everything else is 0), then subtracted from the A matrix, resulting in a new matrix. The only possible way to get the product our new matrix by a vector to equal zero is that the matrix squishes the space into a lesser dimension. Hence, the determinant of it will be zero. Solve for lambda and congratulations you got it. If you applied the same method to 2d rotations lambda will turn out to be imaginary number so no real number eigenvector. Another special case is the shear transformational matrix. It only has X axis as its eigenvector.